

Finite element analysis of dynamic stability of bearingless rotor

WEI Li-jun, LI Shu, LI Xue-chang

(School of Aeronautic Science and Engineering,
Beijing University of Aeronautics and Astronautics, Beijing 100191, China)

Abstract: Dynamic stability equations of bearingless rotor blades were investigated using a simplified model. The aerodynamic loads of blades were evaluated using two-dimensional airfoil theory. Perturbation equations were obtained by linearization of the perturbation. A normal-mode approach was used to transform the equations expressed by nodal degrees of freedom into equations expressed by modal degrees of freedom, which can reduce the dimension of the equations. The stability results of rotor blades were presented using eigenvalue analysis. The shape function matrix was obtained using spline interpolation, which simplified the analysis and made assembly of the inertial matrix, damping matrix, and stiffness matrix a simple mathematical summation. The results indicate that the method is efficient and greatly simplifies the analysis.

Key words: bearingless rotor; dynamic stability; finite element method; spline interpolation; eigenvalue analysis

CLC number: V212.5

Document code: A

Nomenclature

$\mathbf{A}$	Coefficient matrix	L_v, L_w	Aerodynamic force of blade unit length in the lag and flap directions
a	Aerofoil lift force slope	m	Quality of unit length
b_i	i th column of $\mathbf{B}$	m_0	Reference quality of blade unit length
M_{ac}	Aerofoil flexure moment coefficient	M	Torsion in the x direction
C_t	Rotor thrust coefficient	M_ϕ	Torque of blade unit length
c	Aerofoil chord width	$\mathbf{q}$	Global degree of freedom matrix
c_0, c_1	Lift force cipher order coefficient, lift force first-order coefficient	R	Radius of rotor
d_0, d_1	Damping force cipher order coefficient, damping force first-order coefficient	$\mathbf{T}$	Transformation matrix of blade
E	Elasticity modulus	v, w, ϕ	Vibration displacements of lag, flap, and torsion
G	Shear modulus	v_i	Induced inflow velocity
h	Blade element length	V	Relative wind speed of airfoil
I_y	Moment of inertia in the flap direction	V_p, V_t	Components of relative wind speed for airfoil coordination
I_z	Moment of inertia in the lag direction	x, y, z	Three-dimensional orthogonal coordination; x is along the elastic axis before deformation, and y is direct to the blade's leading edge
J	Moment of inertia in the torsion direction	x^i	Coordinate of the i th node
k_a	Aerofoil rotating polar radius		
k_m	Aerofoil rotating mass radius		
k_{m1}, k_{m2}	Constant of airfoil section		

Received: 2013-03-12

Foundation item: National Hi-tech Research and Development Program of China(2012AA112201);

National Natural Science Foundation of China(10772013);

The Fundamental Research Funds for the Central Universities; Aeronautical Science Foundation of China(20100251007)

E-mail: weilijun0318@126.com

β_p	Pre-cone angle	$\theta_{0.75R}$	Angle of attack at 0.75 radius of the blade
θ_i	Blade angle of attack	σ	Rotor solidity
Ω_0	Reference rotor revolution	$\mathbf{H}(x_i)$	Shape function matrix
F	Centrifugal force	$\mathbf{M}(\mathbf{q}), \mathbf{C}(\mathbf{q}), \mathbf{K}(\mathbf{q})$	Inertia, damping, and stiffness matrices of the beam
$\delta U, \delta T, \delta W$	Variations of systemic strain energy, kinetic energy, and virtual work of the outside force	$\mathbf{Q}$	Outside load vector of the beam
$L_{u,nc}, M_{\phi,nc}$	Non-periodic aerodynamic force and moment	$\mathbf{F}$	Transformation matrix of coordinate
ρ	Air density		

The coupled flap-lag-torsion problem plays an important role in the study of the aero-elastic stability of bearingless rotors. Rotor dynamic analysis mainly uses eigenvalue analysis. When establishing the rotor motion equation, the stability is determined by the signs of the eigenvalues^[1]. An important feature of the gyroplane stability problem is the coupling of the trimming stability response and the aero-elasticity^[2-3].

Murty and Raman calculated the nonlinear response of a rotational beam under a certain force by using the finite element method^[4]. Friedmann and Straub calculated the natural vibration performance and stability of coupled flap-lag rotors using the Galerkin finite element method^[5-6]. Sivaneri and Chopra calculated the bearingless and articulated rotor stability and the bearingless rotor aero-elasticity by using the finite element method^[7-9]. Jinseok and Chopra applied this method to the ground and air resonances of a bearingless rotor helicopter^[10-11]. Murthy and Shultz analysed and validated the nonlinear structure item for the redundant load paths of rotor roots, which is essential to the system's dynamic stability^[12].

In the literature, rotor dynamic stability is usually calculated using the finite element method^[13]. The vibration displacement of each element is expressed by Hermite and linear interpolations. The inertia, damping, and stiffness matrices and the outside force vector of beam elements were established and assembled to form the vibration equation of the entire blade^[14]. In this study, the blade was divided into several sections, and each section had three degrees of

freedom: flap, lag, and torsion. The vibration displacement between nodes was calculated using a cubic spline interpolation. This method has the advantage of guaranteeing that the calculation is completed with high precision and the vibration equation of the entire blade is directly obtained. The assembly process of the inertia, damping, and stiffness matrices and external load for the entire blade was simplified into a summation.

1 Dynamic model of rotors

1.1 Physical model and coordinates

When only the vibration characteristics of a rotor system are of concern, the hub centre can be assumed to be fixed, and the rotor blade moves around the hub. Fig. 1 shows the coordinates of the rotor blade.

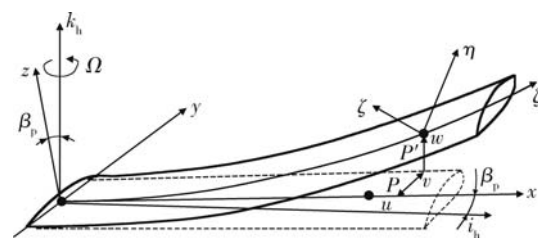


Fig. 1 Coordinates of rotor blade

The coordinates in Fig. 1 are defined as follows:

i_h, j_h, k_h : Fixed coordinates. The origin is at the hub centre, and no rotation occurs.

i_x, j_y, k_z : Rotor rotation coordinates. The origin is at the hub centre; these coordinates rotate around the hub centre along with the rotor, and they are also called the coordinates before blade deformation. The angle between axis i_x and the rotation plane is the pre-cone angle β_p .

i_ξ, j_η, k_ζ : Coordinates after blade deformation. Axis i_ξ coincides with the tangent line of the blade elastic axis after deformation. j_η and k_ζ are vertical with each other in the blade cross

section.

The transform relation from coordinate i_x, j_y, k_z to coordinate i_h, j_h, k_h is as follows:

$$\begin{Bmatrix} i_x \\ j_y \\ k_z \end{Bmatrix} = \begin{bmatrix} \cos \beta_p \cos \psi_m & \cos \beta_p \sin \psi_m & \sin \beta_p \\ -\sin \psi_m & \cos \psi_m & 0 \\ -\sin \beta_p \cos \psi_m & -\sin \beta_p \sin \psi_m & \cos \beta_p \end{bmatrix} \begin{Bmatrix} i_h \\ j_h \\ k_h \end{Bmatrix} = \mathbf{F} \begin{Bmatrix} i_h \\ j_h \\ k_h \end{Bmatrix} \quad (1)$$

The point $P(x, y, z)$ in the elastic coordinates before blade deformation is $P'(\xi, \eta, \zeta)$ in the coordinates after blade deformation. The trans-

form relation from the coordinates before blade deformation to coordinates after blade deformation is as follows:

$$\begin{Bmatrix} i_\xi \\ j_\eta \\ k_\zeta \end{Bmatrix} = \begin{bmatrix} 1 & v' & w' \\ -(v' \cos \theta_1 + w' \sin \theta_1) & \cos \theta_1 & \sin \theta_1 \\ v' \sin \theta_1 - w' \cos \theta_1 & -\sin \theta_1 & \cos \theta_1 \end{bmatrix} \begin{Bmatrix} i_x \\ j_y \\ k_z \end{Bmatrix} = \mathbf{T} \begin{Bmatrix} i_x \\ j_y \\ k_z \end{Bmatrix} \quad (2)$$

1.2 Hamilton's principle

The helicopter blade model used in this study is based on the following assumptions:

- 1) The blades are symmetric aerofoils.
- 2) The axes for the elasticity distortion, centroid, aerodynamic centre, and axial force comprehensive function coincide.
- 3) The blade pre-twist angle is excluded.
- 4) The pre-cone angle β_p is included.

The equations of motion for bearingless rotor blade are obtained using Hamilton's principle

$$\int_{t_1}^{t_2} (\delta U - \delta T - \delta W) dt = 0 \quad (3)$$

The expressions of δU and δT have been defined in Ref. [15] by Hodges and Dowell. Many researchers have contributed to the simplification of these expressions. The basic assumption and non-dimensional normalization are used to obtain further improved expressions as follows:

The variation in strain energy is

$$\begin{aligned} \frac{\delta U}{m_0 \Omega_0^2 R^3} = & \int_0^1 \{ F(v' \delta v' + w' \delta w') + \\ & GJ(\phi' \delta \phi' + \phi' w' \delta v'' + \phi' v'' \delta w' + v'' w' \delta \phi') + \\ & Fk_a^2 \phi' \delta \phi' + [(EI_z \cos^2 v'' + EI_y \sin^2 v'') + \\ & \frac{1}{2}(EI_z - EI_y) w'' \sin 2\theta_1] \delta v' + \\ & [(EI_z \sin^2 w'' + EI_y \cos^2 w'') + \\ & \frac{1}{2}(EI_z - EI_y) v'' \sin 2\theta_1] \delta w'' + \\ & [(EI_z - EI_y) (\frac{1}{2}(w''^2 - v''^2) \sin 2\theta_1 + \\ & v'' w'' \cos 2\theta_1)] \delta \phi \} dx \end{aligned} \quad (4)$$

The variation in kinetic energy is

$$\begin{aligned} \frac{\delta T}{m_0 \Omega_0^2 R^3} = & \int_0^1 m \{ [v + 2\beta_p \dot{w} + \\ & 2 \int_0^x (\dot{v} v' + \dot{w} w') dx - \ddot{v}] \delta v - \\ & [\beta_p (x + 2\dot{v}) + \ddot{w}] \delta w - \\ & [k_m^2 \ddot{\phi} + \frac{1}{2}(k_{m2}^2 - k_{m1}^2) \sin 2\theta_1] \delta \phi \} dx \end{aligned} \quad (5)$$

The virtual work of the applied force is

$$\frac{\delta W}{m_0 \Omega_0^2 R^3} = \int_0^1 (L_v \delta v + L_w \delta w + M_\phi \delta \phi) dx \quad (6)$$

In Eq. (4)–Eq. (6), $(\quad)'$ denotes the partial derivative with respect to the space coordinate x and $(\dot{\quad})$ denotes the partial derivative with respect to the time coordinate t .

In order to simplify the expression, each variable is expressed in its original expression after non-dimensional normalization. For instance, the lag displacement v is still expressed by v instead of v/R .

As shown in Eq. (3)–Eq. (5), $\delta U, \delta T$, and δW are independent of the differential of time for $\delta v, \delta w$, and $\delta \phi$; hence, Hamilton's principle can be written as

$$\Delta = \delta U - \delta T - \delta W = 0 \quad (7)$$

1.3 Aerodynamic force

The aerodynamic force distribution of rotor is calculated from semi-stationary two-dimensional aerofoil theory. Air is assumed to be an incompressible fluid regardless of viscosity, and the inducement inflow is stationary. The aerody-

dynamic force of rotors can be classified into two parts: periodic and non-periodic.

If the comparative wind speed is decomposed into the main axis of the airfoil, as shown in Fig. 2, the two components are

$$V_t = \Omega x \left[\left(1 - \frac{v'^2}{2} \right) \cos \theta_1 - v' w' \sin \theta_1 \right] + \cos \theta_1 \left[\dot{v} - \dot{\phi} \eta_{\frac{3}{4}R} \cos \theta_1 + \Omega(u - v' \eta \cos \theta_1 - w' \eta \sin \theta_1) - \Omega \beta_p (w + \eta \sin \theta_1) \right] + \sin \theta_1 \left[\dot{w} + \dot{\phi} \eta \cos \theta_1 + \Omega \beta_p (v + \eta \cos \theta_1) + v_i \right] + \Omega(v' \cos \theta_1 + w' \sin \theta_1)(v + \eta \cos \theta_1) \quad (8)$$

$$V_p = -\Omega x \left[\left(1 - \frac{v'^2}{2} \right) \sin \theta_1 + v' w' \cos \theta_1 \right] - \sin \theta_1 \left[\dot{v} - \dot{\phi} \eta_{\frac{3}{4}R} \sin \theta_1 + \Omega(u - v' \eta \cos \theta_1 - w' \eta \sin \theta_1) - \Omega \beta_p (w + \eta \sin \theta_1) \right] + \cos \theta_1 \left[\dot{w} + \dot{\phi} \eta \cos \theta_1 + \Omega \beta_p (v + \eta \cos \theta_1) + v_i \right] - \Omega(v' \sin \theta_1 - w' \cos \theta_1)(v + \eta \cos \theta_1) \quad (9)$$

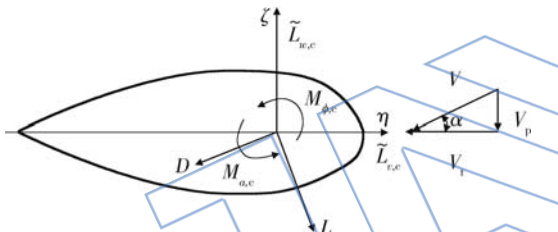


Fig. 2 Aerodynamics of deformed blade section

In Fig. 2, L is lift force; D is damping force. The coordinates of the periodic aerodynamic forces after blade transformation are

$$\tilde{L}_{v,c} = \frac{1}{2} \rho c [-d_0 V_t^2 + (c_0 - d_1) V_t V_p + c_1 V_p^2] \quad (10)$$

$$\tilde{L}_{w,c} = -\frac{1}{2} \rho c [-c_0 V_t^2 + (c_1 + d_0) V_t V_p + d_1 V_p^2] \quad (11)$$

$$M_{\phi,c} = \frac{1}{2} \rho c^2 M_{a,c} (V_t^2 + V_p^2) \quad (12)$$

For the coordinates before blade deformation, the above equations can be rewritten as follows:

$$\begin{bmatrix} L_{v,c} \\ L_{w,c} \end{bmatrix} = \mathbf{T} \begin{bmatrix} \tilde{L}_{v,c} \\ \tilde{L}_{w,c} \end{bmatrix} \quad (13)$$

The transformation matrix^[15] is denoted as Eq. (2). And the non-periodic aerodynamic force^[16] is denoted by

$$L_{w,nc} = \frac{1}{4} \pi \rho c^2 (-\ddot{w} + \Omega x \ddot{\phi} + \frac{1}{4} c \ddot{\phi}) \quad (14)$$

$$M_{\phi,nc} = \frac{1}{4} \pi \rho c^2 \left(\frac{1}{4} c \ddot{w} - \frac{1}{2} c \Omega x \dot{\phi} \right) \quad (15)$$

The induced velocity is obtained by momentum theory. The blade angle of attack is obtained by momentum theory and blade theory

$$\lambda_i = \frac{v_i}{\Omega R} = 1.15 \sqrt{C_t/2} \quad (16)$$

$$\theta_1 = \theta_{0.75R} = 6C_t/(\sigma a + 1.5\lambda_i) \quad (17)$$

$$L_v = L_{v,c} \quad (18)$$

$$L_w = L_{w,c} + L_{w,nc} \quad (19)$$

$$M_\phi = M_{\phi,c} + M_{\phi,nc} \quad (20)$$

1.4 Finite element analysis of rotor

1.4.1 Fundamental principle

Generally, a spring beam simplified from rotors has three directions of vibration: flap, lag, and torsion. The beam is partitioned into several elements. The vibration displacements (and slopes) for the two endpoints of each element are taken as degrees of freedom (Fig. 3). In order to simplify the calculation, rotors are partitioned into n segments, and the length of each segment is h .

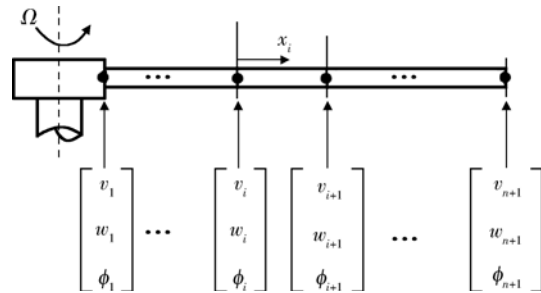


Fig. 3 Global degrees of freedom and local coordinates of an element

Vibration displacement distribution in beam elements is described by Hermite and linear interpolations. The vibration displacement for the i th element (between nodes i and $i+1$) is

$$\begin{bmatrix} v \\ w \\ \phi \end{bmatrix}_i = \mathbf{H}(x_i) \mathbf{q}_i \quad (21)$$

where $\mathbf{q}_i = [v_i \quad v'_i \quad w_i \quad w'_i \quad \phi_i \quad v_{i+1} \quad v'_{i+1} \quad w_{i+1} \quad w'_{i+1} \quad \phi_{i+1}]$, which is the shape function matrix, is described by the interpolation formula.

The following equation can be obtained from Hamilton's principle:

$$\Delta_i = (\delta U - \delta T - \delta W)_i \quad (22)$$

The strain energy, kinetic energy, and virtual work of the outside force of each element are described by the degrees of freedom of each element. Substituting these quantities into Eq. (22) yields

$$\Delta_i = \Delta_i(\ddot{\mathbf{q}}_i, \dot{\mathbf{q}}_i, \mathbf{q}_i, \delta \mathbf{q}_i) \quad (23)$$

The above formula can be rewritten as follows:

$$\Delta_i = \delta \mathbf{q}_i^T \mathbf{M}_i(\mathbf{q}_i) \ddot{\mathbf{q}}_i + \delta \mathbf{q}_i^T \mathbf{C}_i(\mathbf{q}_i) \dot{\mathbf{q}}_i + \delta \mathbf{q}_i^T \mathbf{K}_i(\mathbf{q}_i) \mathbf{q}_i - \delta \mathbf{q}_i^T \mathbf{Q}_i \quad (24)$$

The coefficient matrix of each element can be assembled into a global coefficient matrix. The assembling process is discussed in Ref. [7], and the vibration equation of the spring beam can be written as

$$\mathbf{M}(\mathbf{q}) \ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}) \dot{\mathbf{q}} + \mathbf{K}(\mathbf{q}) \mathbf{q} - \mathbf{Q} = 0 \quad (25)$$

Because of geometric nonlinearities, the mass, damping, and stiffness matrices are dependent on the vector $\mathbf{q}$. $\mathbf{Q}$ is the outside load vector of the beam. The aerodynamic efforts are only of concern for the elements of the outside load vector $\mathbf{Q}$, which contains periodic and non-periodic aerodynamic forces.

1.4.2 Shape function matrix from the spline function

The cubic spline function can be obtained by the method introduced in Ref. [17]. All node vibration displacements, $\mathbf{q} = [v_1 \quad w_1 \quad \phi_1 \quad v_2 \quad w_2 \quad \phi_2 \quad \cdots \quad v_i \quad w_i \quad \phi_i \quad \cdots \quad v_{n+1} \quad w_{n+1} \quad \phi_{n+1}]$, are chosen as the global degrees of freedom.

The vibration displacement of the three vibration degrees of freedom of the i th element of the blade can be expressed by $n+1$ node vibration displacements as follows:

$$\begin{bmatrix} v_i(x) \\ w_i(x) \\ \phi_i(x) \end{bmatrix} = \begin{bmatrix} \mathbf{sv}_i(x) \\ \mathbf{sw}_i(x) \\ \mathbf{s\phi}_i(x) \end{bmatrix} \mathbf{q} \quad (26)$$

The row vectors in Eq. (26) are

$$\begin{aligned} \mathbf{sv}_i(x) &= \frac{(x-x^i)^2}{h^2} \left(1 + 2 \frac{x-x^{i-1}}{h}\right) \mathbf{e}_{3i-2}^T + \\ &\quad \frac{(x-x^{i-1})^2}{h^2} \left(1 + 2 \frac{x^i-x}{h}\right) \mathbf{e}_{3i-5}^T + \\ &\quad \frac{(x-x^i)^2(x-x^{i-1})}{h^2} \mathbf{Av}_{i-1} + \\ &\quad \frac{(x-x^{i-1})^2(x-x^i)}{h^2} \mathbf{Av}_i \end{aligned} \quad (27)$$

$$\begin{aligned} \mathbf{sw}_i(x) &= \frac{(x-x^i)^2}{h^2} \left(1 + 2 \frac{x-x^{i-1}}{h}\right) \mathbf{e}_{3i-1}^T + \\ &\quad \frac{(x-x^{i-1})^2}{h^2} \left(1 + 2 \frac{x^i-x}{h}\right) \mathbf{e}_{3i-4}^T + \\ &\quad \frac{(x-x^i)^2(x-x^{i-1})}{h^2} \mathbf{Aw}_{i-1} + \\ &\quad \frac{(x-x^{i-1})^2(x-x^i)}{h^2} \mathbf{Aw}_i \end{aligned} \quad (28)$$

$$\begin{aligned} \mathbf{s\phi}_i(x) &= \frac{(x-x^i)^2}{h^2} \left(1 + 2 \frac{x-x^{i-1}}{h}\right) \mathbf{e}_{3i}^T + \\ &\quad \frac{(x-x^{i-1})^2}{h^2} \left(1 + 2 \frac{x^i-x}{h}\right) \mathbf{e}_{3i-3}^T + \\ &\quad \frac{(x-x^i)^2(x-x^{i-1})}{h^2} \mathbf{A\phi}_{i-1} + \\ &\quad \frac{(x-x^{i-1})^2(x-x^i)}{h^2} \mathbf{A\phi}_i \end{aligned} \quad (29)$$

In Eq. (29), x_i is the coordinate of the i th node. $\mathbf{e}$ is a vector, which consists of $(3n+3)$ dimensions. $\mathbf{e}^T$, which is the superscript of vector $\mathbf{e}$, represents a transposed matrix. Here, $\mathbf{e}$ is a column vector; therefore, $\mathbf{e}^T$ is a row vector. Subscripts represent node coordinates; for example, $\mathbf{e}_{3i}$ means the 3 i th element of $\mathbf{e}$. $\mathbf{Av}_i$ is the i th row of the matrix $\mathbf{Av}$. The rest are similar.

In Eq. (27)–Eq. (29), $i = 2, 3, \dots, n+1$.

The boundary condition of the blade is as follows:

Built-in root segment

$$\begin{aligned} v &= 0 & w &= 0 & \phi &= 0 \\ v' &= 0 & w' &= 0 & \phi' &= 0 \end{aligned}$$

Free blade tip

$$v'' = 0 \quad w'' = 0 \quad \phi'' = 0$$

Based on the three slope criteria,

$$\mathbf{B} = \frac{3}{2h} \begin{bmatrix} 2 & 0 & & & & \\ 0.5 & 2 & 0.5 & & & \\ & 0.5 & 2 & \ddots & & \\ & & \ddots & \ddots & 0.5 & \\ & & & 0.5 & 2 & 0 \\ & & & & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & & & & \\ 1 & 0 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & 1 & 0 & 1 & \\ & & & 1 & 0 & 0 \\ & & & & -1 & 1 \end{bmatrix} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \quad \cdots \quad \mathbf{b}_n \quad \mathbf{b}_{n+1}]$$

$\mathbf{B}$ is a $(n+1) \times (n+1)$ square matrix.

Let

$$\begin{aligned} \mathbf{A}\mathbf{v} &= [\mathbf{b}_1 \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{b}_2 \quad \mathbf{0} \quad \mathbf{0} \quad \cdots \quad \mathbf{b}_n \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{b}_{n+1} \quad \mathbf{0} \quad \mathbf{0}] \\ \mathbf{A}\mathbf{w} &= [\mathbf{0} \quad \mathbf{b}_1 \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{b}_2 \quad \mathbf{0} \quad \cdots \quad \mathbf{0} \quad \mathbf{b}_n \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{b}_{n+1} \quad \mathbf{0}] \\ \mathbf{A}\boldsymbol{\phi} &= [\mathbf{0} \quad \mathbf{0} \quad \mathbf{b}_1 \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{b}_n \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{b}_{n+1}] \end{aligned}$$

In the three equations above, $\mathbf{A}\mathbf{v}$, $\mathbf{A}\mathbf{w}$, and $\mathbf{A}\boldsymbol{\phi}$ are $(n+1) \times (3n+3)$ matrices.

Substituting Eq. (26) – Eq. (29) into Eq. (22) yields

$$\Delta_i = \delta \mathbf{q}^T \mathbf{M}_i(\mathbf{q}) \ddot{\mathbf{q}} + \delta \mathbf{q}^T \mathbf{C}_i(\mathbf{q}) \dot{\mathbf{q}} + \delta \mathbf{q}^T \mathbf{K}_i(\mathbf{q}) \mathbf{q} - \delta \mathbf{q}^T \mathbf{Q}_i \quad (30)$$

In the entire beam, inertia matrix is

$$\mathbf{M}(\mathbf{q}) = \sum_{i=1}^n \mathbf{M}_i(\mathbf{q})$$

Damping matrix is

$$\mathbf{C}(\mathbf{q}) = \sum_{i=1}^n \mathbf{C}_i(\mathbf{q})$$

Stiffness matrix is

$$\mathbf{K}(\mathbf{q}) = \sum_{i=1}^n \mathbf{K}_i(\mathbf{q})$$

Outside load vector is

$$\mathbf{Q} = \sum_{i=1}^n \mathbf{Q}_i$$

The boundary condition is built-in, and the following blade vibration equation can finally be obtained as Eq. (25). In Eq. (25), $\mathbf{q}$ is the $3n \times 1$ column vector.

2 Solution procedure

2.1 Steady-state response

Before solving for the stability, the blade steady-state response in hovering state should be solved first. Once the item with respect to time in the equations is omitted, the steady-state response equation can be written as

$$\mathbf{K}(\mathbf{q}_0) \mathbf{q}_0 = \mathbf{Q}_0 \quad (31)$$

Eq. (31) is a non-linear, non-homogeneous equation, and the approximate solution can be

obtained by numerical methods. Newton's method was applied in this study. The initial iteration value is the solution of the corresponding homogeneous equation $\mathbf{q}_0^{(0)} = \mathbf{0}$. The vector $\mathbf{q}_0$ contains only the periodic aerodynamic efforts. Theoretically, Eq. (30) has other solutions. The solution obtained from $\mathbf{q}_0 = \mathbf{0}$ is the solution closest to $\mathbf{q}_0 = \mathbf{0}$. The termination condition for the iteration is $\|\mathbf{K}\mathbf{q} - \mathbf{Q}\|_2 < T_{ol}$ (T_{ol} is the given tolerance limit).

2.2 Linear perturbation movement equation

Assume a perturbation vibration $\Delta \mathbf{q}$ near the steady-state response; that is, assume $\mathbf{q} = \mathbf{q}_0 + \Delta \mathbf{q}$. Substituting this into Eq. (25) and simplifying yields the linear perturbation movement equation

$$\mathbf{M}(\mathbf{q}_0) \Delta \ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}_0) \Delta \dot{\mathbf{q}} + \mathbf{K}(\mathbf{q}_0) \Delta \mathbf{q} = \mathbf{0} \quad (32)$$

2.3 Natural mode characteristics and mode conversion

The damping item $\mathbf{C}(\mathbf{q}_0) \Delta \dot{\mathbf{q}}$ in Eq. (32) is omitted and the aerodynamic force is neglected to obtain the control equation for the blade's natural vibration

$$\mathbf{M}(\mathbf{q}_0) \ddot{\mathbf{q}} + \mathbf{K}(\mathbf{q}_0) \mathbf{q} = \mathbf{0} \quad (33)$$

The coefficient matrix of this equation is symmetric, and all eigenvalues are real; this is the natural frequency of flapping, lag, and torsion. The corresponding eigenvectors are natural modes.

In order to limit the degrees of freedom when the system stability is calculated, the mode transition method is needed to transform

the global degrees of freedom into mode space. Take the first N eigenvectors of Eq. (33) to form the mode space Φ ($3n \times N$ orders), and let

$$\Delta q = \Phi p \quad (34)$$

This is substituted into Eq. (32) and multiplied with Φ^T to obtain

$$M\ddot{p} + C\dot{p} + Kp = 0 \quad (35)$$

where

$$M = \Phi^T M(q_0) \Phi$$

$$C = \Phi^T C(q_0) \Phi$$

$$K = \Phi^T K(q_0) \Phi$$

We can transform Eq. (35) into the following state equation:

$$\dot{z} = Az \quad (36)$$

where

$$z = \begin{bmatrix} \dot{p} & p \end{bmatrix}^T$$

$$A = \begin{bmatrix} -M^{-1}C & -M^{-1}K \\ I_{N \times N} & 0_{N \times N} \end{bmatrix}$$

Eigenvalue λ of the coefficient matrix A is a complex number. The real part of the eigenvalue represents system mode damping. If the real part is negative, the system is stable. Otherwise, it is unstable. The imaginary part represents the frequency of the system mode.

3 Numerical results

Structural and inertia parameters ($E_{Iy}, E_{Iz}, G_J, k_m, k_a$, etc.) determine the blade's natural frequency, which is commonly designed as the blade configuration parameters. Generally, the natural frequency can not explicitly express the structural and inertia parameters. Accordingly, when the natural frequency is determined, these parameters are commonly obtained using the iteration method^[18]. The detailed procedure for obtaining the structural and inertia parameters with the natural frequency is documented in Ref. [18].

In this study, a bearingless rotor blade was used as an example of the numerical approach. The fundamental natural vibration frequencies with respect to the three vibration modes of blade model are $\omega_v = 1.15$, $\omega_w = 1.5$, and $\omega_\phi = 2.5$. The basic parameters of the blade are listed

in Table 1 and Table 2. All the parameters in the tables are numerical values after non-dimensional normalization.

Table 1 Structural and inertial parameters of a bearingless rotor

Parameter	Numerical value
$E_{Iy}/(m_0 \Omega_0^2 R^3)$	0.014 486
$E_{Iz}/(m_0 \Omega_0^2 R^3)$	0.166 908
$G_J/(m_0 \Omega_0^2 R^3)$	0.000 925
k_{m1}/R	0
k_{m2}/R	0.025
k_a	0.037 5

Table 2 Geometry and aerodynamic parameters of a bearingless rotor

Parameter	Numerical value
c/R	$\pi/40$
β_p	0.05
a	6
M_{ac}	0
c_0	0
c_1	6
d_0	0.009 5
d_1	0

A comparison of data from Table 3 with the results indicated in Ref. [7] indicates that the present results agree well with the theoretical values.

Table 4 summarizes the numerical results for the fundamental frequency in natural order when different element numbers are used. When the element number was 30 or 25, the relative differences in the results were less than 0.05% for all three vibration directions.

Increasing the rotor speed tended to gradually decrease the lag and flapping displacements of the blade tip's steady-state response; the torsion displacement was rapidly changed into the other direction and then diminished gradually, as shown in Fig. 4.

Table 3 Data comparison of fundamental natural frequencies

Vibration direction	Method	Numerical value	Relative error/%
Lag	Theoretical values	1.15	
	Ref. [7]	1.1211	2.51
	Method in this paper	1.1286	1.87
Flap	Theoretical values	1.5	
	Ref. [7]	1.5181	1.21
	Method in this paper	1.4969	0.21
Torsion	Theoretical values	2.5	
	Ref. [7]	2.4724	1.10
	Method in this paper	2.5525	2.10

Table 4 Influence of element number on fundamental frequencies

Element number	Vibration direction		
	Lag	Flap	Torsion
15	1.1269	1.4924	2.5524
20	1.1278	1.4949	2.5523
25	1.1283	1.4962	2.5524
30	1.1286	1.4969	2.5525
35	1.1288	1.4974	2.5526
40	1.1289	1.4977	2.5527

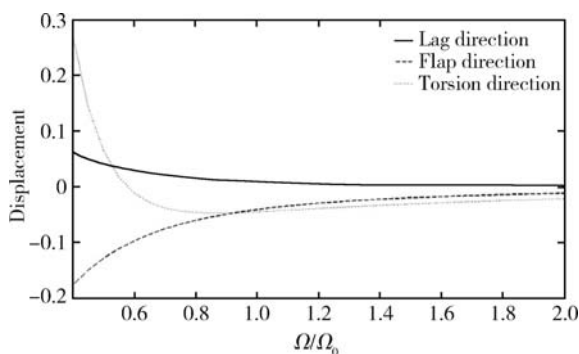
**Fig. 4** Steady deflections of a blade tip at different rotor speeds

Table 5 lists data for the stable response of the $(n+1)$ th node obtained from Eq. (31): the lag, flap, and torsion displacements at the blade tip. The termination criterion for the iteration procedure is $\| \mathbf{Kq} - \mathbf{Q} \|_2 < 10^{-8}$.

Table 5 Steady deflections of the blade tip

Element number	Vibration direction		
	Lag	Flap	Torsion
15	0.00922	-0.0409	-0.0448
20	0.00921	-0.0408	-0.0448
25	0.00920	-0.0408	-0.0448
30	0.00920	-0.0408	-0.0448
35	0.00919	-0.0408	-0.0448
40	0.00919	-0.0408	-0.0448

When the element number was 30, the stable response (Table 5) and fundamental frequencies in nature order (Table 4) at the blade tip were all highly precise. Considering the amount and time of calculation, the element number was set to be 30 for the following numerical calculations.

In mode conversion, numbers of natural mode influence the precision of the stability results. Table 6 lists the real parts of the eigenvalue of the fundamental vibration mode in three vibration directions when different numbers of modes were used. As shown in the data of Table 6, the stability results were already highly precise when the first six-order natural modes were chosen. The fundamental mode of vibration was stable in the flap and torsion directions of this rotor model and slightly unstable in the lag direction. The stability can be ensured by changing rotor structure parameters.

The fundamental mode of vibration has the most influence on the system vibration. When the rotation speed changes, the stability of the fundamental mode changes too. As shown in Fig. 5, even if the rotation speed increases, the fundamental mode of vibration in the torsion direction stays in the stable section at all times. In

the flap direction, it is stable in the medium speed section but unstable in the low and high speed sections. In the lag direction, it is unstable at all times, but the stability tends to decrease.

Table 6 Real parts of eigenvalues of blade fundamental modes

Number of modes	Flap direction	Lag direction	Torsion direction
4	-0.00101	0.006923	-0.13862
5	-0.00101	0.006923	-0.13860
6	-0.00093	0.006924	-0.13855
7	-0.00093	0.006924	-0.13855
8	-0.00093	0.006924	-0.13855

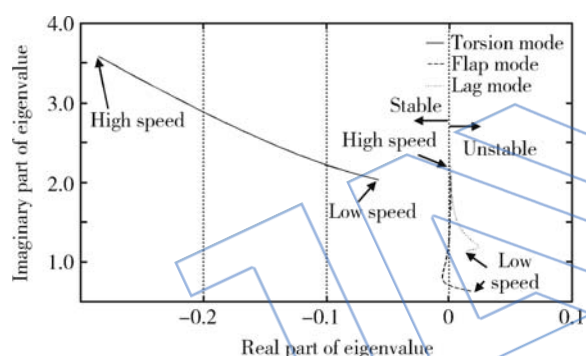


Fig. 5 Locus of three fundamental modes when rotor speed is increased

Since the ground and air resonances of a helicopter are mainly caused by the coupling of the rotor lead-lag mode and the fuselage vibration mode, a bearingless rotor blade is usually tender.

4 Conclusion

The cubic spline interpolation function has consistent first-order and second-order derivatives. The shape function matrix can be obtained by cubic spline interpolation in finite element analysis of the rotor vibration characteristics of a helicopter to simplify the analysis process.

The results obtained in this study are consistent with those obtained in the existing literature. The feasibility of all the methods was vali-

dated using a simplified theoretical analysis.

The finite element method can be easily extended toward calculating the dynamic stability of articulated and bearingless rotors.

References:

- [1] ZHANG Xiaogu. Helicopter dynamics design[M]. Beijing: Aviation Industry Press, 1995: 74-79. (in Chinese)
- [2] Friedmann P P. Rotory-wing aeroelastica historical perspective and some current trends[R]. Norfolk, Virginia: 44th AIAA/ASME/ASCE/AHS Structures, Structural Dynamics, and Materials Conference, 2003.
- [3] Fridemann P P. Rotory-wing aeroelastic; current status and future trends[J]. AIAA Journal, 2004, 42(10): 1953-1972.
- [4] Murty A V, Raman A. Non-linear dynamic analysis of rotors by finite element methods[J]. Journal of Sound and Vibration, 1980, 69(4): 559-568.
- [5] Friedmann P P, Straub F K. Application of the finite element method to rotary wing aeroelasticity[J]. Journal of the American Helicopter Society, 1980, 25(3): 36-44.
- [6] Straub F K, Friedmann P P. A galerkin type finite element for rotary-wing aeroelastic[J]. Hover and Forward Flight, 1981, 5(5): 75-98.
- [7] Sivaneri N T, Chopra I. Dynamic stability of a rotor blade using finite element analysis[J]. AIAA Journal, 1982, 20(5): 716-723.
- [8] Sivaneri N T, Chopra I. Finite element analysis for bearingless rotor blade aeroelasticity[J]. Journal of American Helicopter Society, 1984, 29(2): 42-51.
- [9] Sivaneri N T. Aeroelastic stability of rotor blades using finite element analysis[D]. Palo Alto County, California: Stanford University, 1982.
- [10] Jinseok J, Chopra I. Ground and air resonance of bearingless rotors in hover[R]. AIAA 87-0924, 1987.
- [11] Jinseok J. Ground and air resonance of bearingless rotors in hover and forward flight[D]. Maryland: University of Maryland College Park, 1988.
- [12] Murthy V R, Shultz L A. Linear and nonlinear dynamic analysis of redundant load path bearingless rotor systems[R]. NASA CR-197220, 1994.
- [13] WANG Yifeng, WANG Haowen, GAO Zheng. Dynamic modeling of helicopter rotor blades[J]. Tsinghua Science and Technology, 2009, 14(suppl. 2): 84-88. (in Chinese)
- [14] In-Gyu L, In L. Aeroelastic analysis of bearingless rotors with a composite flexbeam[J]. Composite Structure, 2009, 88: 570-578.
- [15] Hodges D H, Dowell E H. Nonlinear equations of motion for elastic bending and torsion of twisted nonuniform Blades[R]. NASA TN D-7818, 1974.
- [16] Fung Y C. An introduction to the theory of aeroelasticity

- [M]. New York: Dover Publication, 1969.
- [17] LI Qingyang, WANG Nengchao, YI Dayi. Numerical analysis[M]. 4th ed. Beijing: Tsinghua University Press, 2001: 51-55. (in Chinese)
- [18] Hodges D H, Ormiston R A. Stability of elastic bending and torsion of uniform cantilever rotor blades in hover with variable structural coupling[R]. NASA TN D-8192, 1976.

JALSP